

Infinite symmetric group and combinatorial descriptions of semigroups of double cosets

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We show that double cosets of the infinite symmetric group with respect to some special subgroups admit natural structures of semigroups. We interpret elements of such semigroups in combinatorial terms (chips, colored graphs, two-dimensional surfaces with polygonal tiling) and describe explicitly the multiplications.

The purpose of the paper is to formulate several facts of the representation theory of infinite symmetric groups (some references are [35], [36], [15], [26], [31], [25], [11], [19]) in a natural generality. We consider the infinite symmetric group \mathbb{S}_∞ , its finite products with itself $\mathbb{S}_\infty \times \cdots \times \mathbb{S}_\infty$, and some wreath products. Let G, K be groups of this kind.

Sometimes double cosets $K \backslash G / K$ admit natural associative product (this is a phenomenon, which is usual for infinite dimensional groups²). We formulate wide sufficient conditions for existence of such multiplications. Our main purpose is to describe such semigroups (in fact, categories), in a strange way this can be obtained in a very wide generality.

Note that a link between symmetric groups and two-dimensional surfaces arises at least to Hurwitz [8], see a recent discussion in [14], see also below Subsection 9.6. Categories of polygonal bordisms were discussed in [2], [16], [19]. Note that they can be regarded of combinatorial analogs of 'conformal field theories', see [34], [18].

It seems that our constructions can be interesting for finite symmetric groups, we give descriptions of various double cosets spaces and also produce numerous 'parameterizations' of symmetric groups.

1 Introduction

1.1. Double cosets. Let G be a group, K, L be its subgroup. A *double coset* on G is a set of the form KgL , where g is a fixed element of G . By $K \backslash G / L$ we denote the space of all double cosets.

1.2. Convolution of double cosets. Let G be a Lie group, K a compact subgroup. Denote by $\mathcal{M}(K \backslash G / K)$ the space of all compactly supported charges (signed measures) on G invariant with respect to left and right shifts by to elements of K . The space $\mathcal{M}(K \backslash G / K)$ is an algebra with respect to the convolution $\mu * \nu$ on G . Denote by $\Pi \in \mathcal{M}(K \backslash G / K)$ the probability Haar measure on K . For $\mu \in \mathcal{M}(K \backslash G / K)$, we have $\Pi * \mu * \Pi = \mu$.

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²For compact groups double cosets form so-called "hypergroups", convolution of a pair of double cosets is a measure on $K \backslash G / K$.

Let ρ be a unitary representation of G in a Hilbert space H . Denote by $H^{[K]}$ the set of all K -fixed vectors in H . For a measure $\mu \in \mathcal{M}(K \setminus G/K)$ consider the operator $\rho(\mu)$ in H given by

$$\rho(\mu) = \int_G \rho(g) d\mu.$$

By definition,

$$\rho(\mu)\rho(\nu) = \rho(\mu * \nu).$$

Next, $\rho(\Pi)$ is the operator $P^{[K]}$ of projection to the subspace $H^{[K]}$. Evidently, for $\mu \in \mathcal{M}(K \setminus G/K)$, we have

$$\rho(\mu) = \rho(\Pi * \mu * \Pi) = P^{[K]}\rho(\mu)P^{[K]}.$$

Therefore, the matrix of $\rho(\mu)$ with respect to the orthogonal decomposition $H = H^{[K]} \oplus (H^{[K]})^\perp$ has the form $\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$. In other words, we can regard the operator $\rho(\mu)$ as an operator

$$\rho(\mu) : H^{[K]} \rightarrow H^{[K]}.$$

In some special cases (G is a semisimple Lie group, K is the maximal compact subgroup; G is a p -adic semisimple group, K is the Iwahori subgroup, etc.) the algebra $\mathcal{M}(K \setminus G/K)$ and its representation in $H^{[K]}$ are widely explored in representation theory.

1.3. Noncompact subgroups. Now let a subgroup K be noncompact. Then the construction breaks in both points. First, there is no nontrivial finite K -biinvariant measures on G . Second, for faithful unitary representations of G the subspace $H^{[K]}$ is trivial.

However, for infinite dimensional groups these objects appear again in another form.

1.4. Multiplication of double cosets. For some infinite-dimensional groups G and for some subgroups K the following facts (the “*multiplicativity theorem*”) hold:

- There is a natural associative multiplication $\mathfrak{g} \circ \mathfrak{h}$ on the set $K \setminus G/K$.
- Let $\mathfrak{g} \in K \setminus G/K$ and $g \in \mathfrak{g}$. We define the operator

$$\rho(\mathfrak{g}) := P^{[K]}\rho(g) : H^{[K]} \rightarrow H^{[K]}$$

Then

$$\rho(\mathfrak{g})\rho(\mathfrak{h}) = \rho(\mathfrak{g} \circ \mathfrak{h}).$$

Multiplicativity theorems were firstly observed by Ismagilov (see [9], [10], see also [24]) for $G = SL(2, Q)$, $K = SL(2, Z)$, where Q is complete non-locally compact non-archimedean field, and $Z \subset Q$ is the ring of integer elements.

Olshanski used semigroups of double cosets in the representation theory of infinite-dimensional classical groups [27]–[30] and the representation theory

of infinite symmetric groups in [31], see also [32]. In [17] this approach was extended to groups of automorphisms of measure spaces.

In [18] it was given a simple unified proof of multiplicativity theorems, which covered all known cases. This also implied that the phenomenon holds under quite weak conditions for a pair $G \supset K$. However it was not clear how to describe double cosets $K \backslash G/K$ and their products explicitly.

Another standpoint of this paper was a work of Nessonov [22]–[23]. For $G = \mathrm{GL}(\infty, \mathbb{C}) \times \cdots \times \mathrm{GL}(\infty, \mathbb{C})$ (number of factors is arbitrary) with the diagonal subgroup $K = \mathrm{U}(\infty)$, he classified all K -spherical functions on the group G with respect to K .

In [20] and [21] it was described multiplication of double cosets for a wide class of pairs $G \supset K$ of infinite-dimensional classical groups. In [19] parallel description was obtained for the pair $G = \mathbb{S}_\infty \times \mathbb{S}_\infty \times \mathbb{S}_\infty$, $K = \mathbb{S}_{\infty-n}$.

The purpose of this paper is to obtain a similar description for multiplication of double cosets for general pairs related to infinite symmetric groups.

2 (G, K) -pairs and their trains. Definitions and a priory theorems

First, we define some pairs $G \supset K$ of groups, for which multiplicativity theorems hold.

Secondly, we formulate a priory theorems on the semigroups and categories of double cosets. Proofs are omitted because they are one-to-one copies of proofs given in [21] for infinite-dimensional classical groups.

2.1. Notation.

- \mathbb{N} are natural numbers.
- $\mathbb{M}(\alpha)$ is the initial segment $\{1, 2, \dots, \alpha\}$ of \mathbb{N} .
- $\mathbb{N}_1, \mathbb{N}_2, \dots$ are disjoint copies of the set \mathbb{N} .
- $\mathbb{M}_j(\alpha)$ are initial segments $\{1, 2, \dots, \alpha\}$ of \mathbb{N}_j .
- $\mathbb{I}(\zeta)$ are finite sets with ζ elements.
- \sqcup, \coprod are symbols for disjoint union of sets.
- S_n is the finite symmetric group.

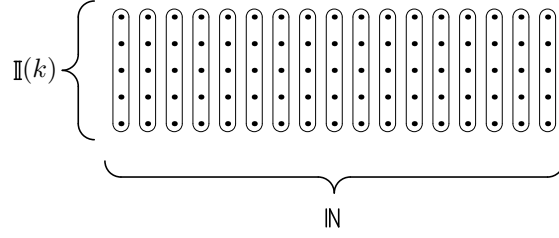
2.2. Infinite symmetric group. We denote by \mathbb{S}_∞ the group of finite permutations of \mathbb{N} . We say that a permutation σ is *finite*, if $\sigma(i) = i$ for all but finite number of i . Denote by $\mathrm{supp}(\sigma)$ the *support* of σ , i.e., the set of $i \in \mathbb{N}$ such that $\sigma(i) \neq i$.

For a countable set Ω denote by $\mathbb{S}_\infty(\Omega)$ the group of all finite permutations of Ω .

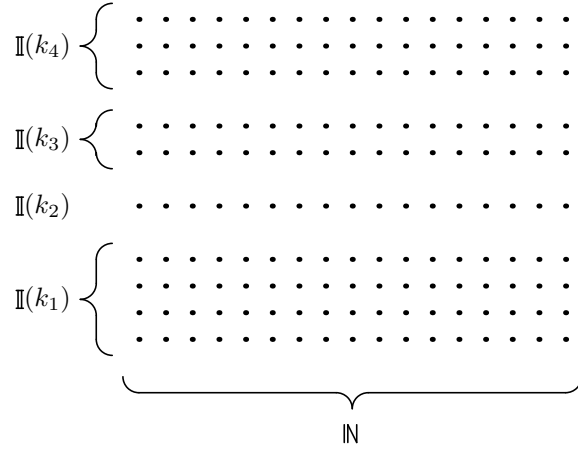
We represent permutations by infinite 0-1 matrices.

By $\mathbb{S}_\infty^\alpha \subset \mathbb{S}_\infty$ we denote the group of permutations having the form

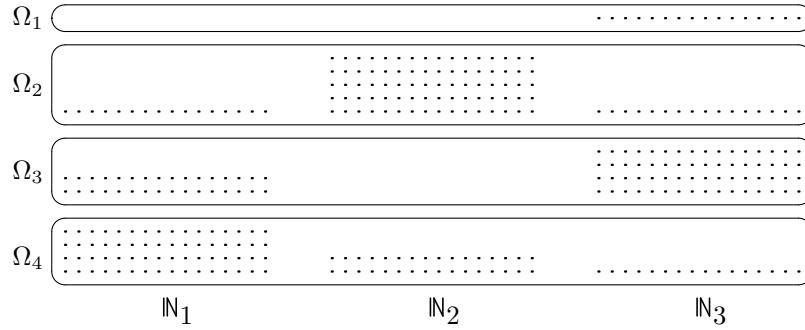
$$\sigma = \begin{pmatrix} 1_\alpha & 0 \\ 0 & * \end{pmatrix},$$



a) The group $\mathbb{S}_\infty \ltimes (S_k)^\infty$. The group \mathbb{S}_∞ acts by permutations of columns. The normal divisor $(S_k)^\infty$ permute elements in each column. The semi-direct product consists of permutations preserving partition of the strip into columns



b) The group $\mathbb{S}_\infty \ltimes (S_{k_1} \times \dots \times S_{k_p})^\infty$. The \mathbb{S}_∞ acts by permutations of columns. The normal divisor acts by permutations inside each sub-column.



c) The set $\cup_{i=1}^p \cup_{j=1}^q (\mathbb{N}_i \times \mathbb{I}(\zeta_{ji}))$. Here $q = 4$, $p = 3$, $Z = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 5 & 1 \\ 2 & 0 & 4 \\ 4 & 2 & 1 \end{pmatrix}$

Figure 1: Reference to Section 2.

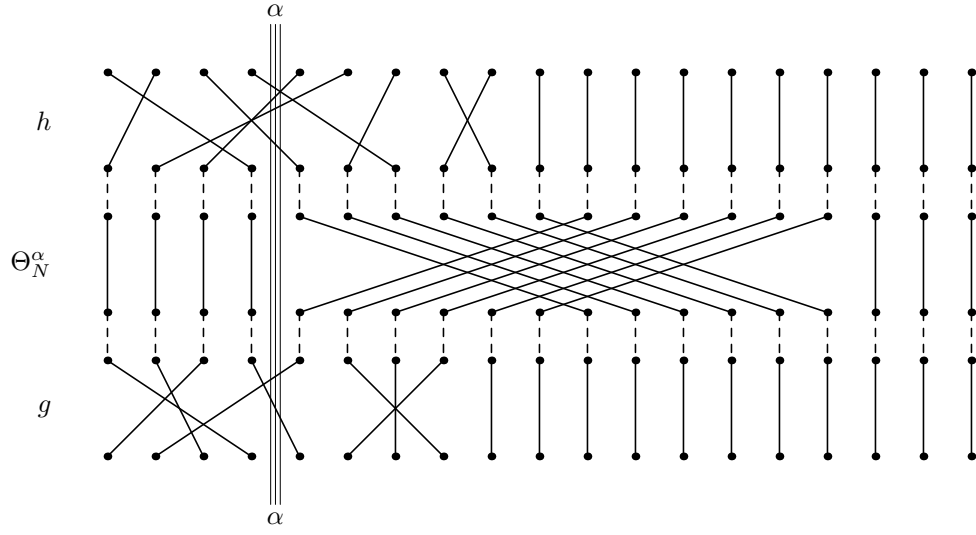


Figure 2: Reference to Subsection 2.13. *Forcing apart*. We draw an element of a symmetric group as a collection of inclined lines. The product is the connection of the corresponding ends. Here $\alpha = \beta = \gamma$. If N is large, the substitution Θ_N^α sent the set $\text{supp}(h) \cap \{k > \alpha\}$ outside $\text{supp}(g)$.

where 1_α is the unit $\alpha \times \alpha$ matrix.

Let $K = \mathbb{S}_\infty(\mathbb{N}_1) \times \cdots \times \mathbb{S}_\infty(\mathbb{N}_p)$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_p)$ we define the subgroup $K^\alpha \subset K$,

$$K^\alpha := \mathbb{S}_\infty^{\alpha_1}(\mathbb{N}_1) \times \cdots \times \mathbb{S}_\infty^{\alpha_p}(\mathbb{N}_p).$$

2.3. Topological infinite symmetric group. We denote by \mathbf{S}_∞ the group of all permutations of \mathbb{N} . Define the subgroups $\mathbf{S}_\infty^\alpha \subset \mathbf{S}_\infty$ as above. Define the topology on \mathbf{S}_∞ assuming that \mathbf{S}_∞^α form a fundamental systems of open neighborhoods of unit. In other words, a sequence σ_j converges to σ if for any $k \in \mathbb{N}$ we have $\sigma_j k = \sigma k$ for sufficiently large j . The group \mathbf{S}_∞ is a totally disconnected topological group.

The classification of irreducible unitary representations of \mathbf{S}_∞ was obtained by Lieberman [15], see expositions in [26], [18].

Theorem 2.1 *a) Each irreducible representation of \mathbf{S}_∞ is induced from a finite-dimensional representation $\tau \otimes id$ of a subgroup $S_\alpha \times \mathbf{S}_\infty^\alpha$.*

b) Any unitary representation of \mathbf{S}_∞ is a direct sum of irreducible representations.

Note that the quotient space $\mathbb{S}_\infty/\mathbb{S}_\infty^\alpha = \mathbf{S}_\infty/\mathbf{S}_\infty^\alpha$ is countable, and therefore the definition of induced representations survives (see, e.g., [13], 13.2).

In a certain sense, the Lieberman theorem opens and closes the representation theory of the group \mathbf{S}_∞ . However it is an important element of wider theories.

2.4. Reformulations of continuity. Let ρ be a unitary representation of \mathbb{S}_∞ in a Hilbert space H . Denote by $H^\alpha \subset H$ the subspace of all \mathbb{S}_∞^α -fixed vectors. We say that a representation ρ is *admissible* if $\cup H^\alpha$ is dense in H .

Denote by \mathbf{B}_∞ the semigroup of matrices composed of 0 and 1 such that each row and each column contains ≤ 1 units. We equip \mathbf{B}_∞ with the topology of element-wise convergence, the group \mathbb{S}_∞ is dense in \mathbf{B}_∞ .

Theorem 2.2 (see [26], [18]) *The following conditions are equivalent:*

- 1) ρ is continuous in the topology of \mathbf{S}_∞ ;
- 2) ρ is admissible;
- 3) ρ admits a continuous extension to the semigroup \mathbf{B}_∞ .

This statement has a straightforward extension to products of symmetric groups $\mathbf{K} = \mathbf{S}_\infty \times \cdots \times \mathbf{S}_\infty$

2.5. Wreath products. Let U be a finite group. Consider the countable direct product $\mathbf{U}^\infty := U \times U \times U \dots$, it is the group, whose elements are infinite sequences (u_1, u_2, \dots) . Consider also the restricted product U^∞ , whose elements are sequences such that $u_j = 1$ starting some place. The group \mathbf{U}^∞ is equipped with the topology of direct product, the group U^∞ is discrete.

Permutations of sequences (u_1, u_2, \dots) induce automorphisms of U^∞ and \mathbf{U}^∞ . Consider the semidirect products $K := \mathbb{S}_\infty \ltimes U^\infty$ and $\mathbf{K} := \mathbf{S}_\infty \ltimes \mathbf{U}^\infty$, see, e.g., [13], 2.4, they are called *wreath products* of \mathbb{S}_∞ and U .

EXAMPLE. The infinite *hyperoctahedral group* is a wreath product of \mathbb{S}_∞ and \mathbb{Z}_2 . \square

Our main example is the wreath product of \mathbb{S}_∞ and a finite symmetric group S_k . We realize it as a group of finite permutations of the $\mathbb{N} \times \mathbb{I}(k)$. The group \mathbb{S}_∞ acts by permutations of \mathbb{N} , and subgroups $S_k \subset (S_k)^\infty$ act by permutations of sets $\{m\} \times \{1, \dots, k\}$, see Figure 1.a.

2.6. Representations of wreath products. For $K = \mathbb{S}_\infty \ltimes U^\infty$ we define a subgroup K^α as the semidirect product of \mathbb{S}_∞^α and the subgroup

$$U^{\infty-\alpha} := \underbrace{1 \times \dots \times 1}_{\alpha \text{ times}} \times U \times U \times \dots$$

We define *admissible representations* as above. Again, a unitary representation of $K = \mathbb{S}_\infty \ltimes U^\infty$ is admissible if and only if it is continuous in the topology of the group $\mathbf{K} = \mathbf{S}_\infty \ltimes \mathbf{U}^\infty$.

2.7. (G, K) -pairs, see Figure 1. Fix positive integers q, p and $q \times p$ -matrix

$$Z := \{\zeta_{ji}\}$$

consisting of non-negative integers. Assume that it has no zero columns and no zero rows.

Fix a collection $\Lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_q \end{pmatrix}$ of nonnegative integers. Fix sets L_1, \dots, L_q , such that L_j has λ_j elements. Consider the collection of sets

$$\mathbb{N}_i \times \mathbb{I}(\zeta_{ij}).$$

Assume that Ω_j is the disjoint union

$$\Omega_j := L_j \sqcup \coprod_{i \leq p} (\mathbb{N}_i \times \mathbb{I}(\zeta_{ji}))$$

and assume that all sets Ω_j obtained in this way are mutually disjoint. Set

$$G := G[Z, \Lambda] = \prod_{j=1}^q \mathbb{S}_\infty(\Omega_j).$$

Next, set

$$K^\circ = K^\circ[Z] := \prod_{i=1}^p \left(\mathbb{S}_\infty(\mathbb{N}_i) \ltimes \left(\prod_j S_{\zeta_{ij}} \right)^\infty \right).$$

We have a tautological embedding $K^\circ \rightarrow G$.

Also, we set

$$K^{\otimes}[Z] := \prod_{i=1}^p \mathbb{S}_{\infty}(\mathbb{N}_i) \subset K^{\circ}[Z].$$

Below (G, K) denotes a pair (group, subgroup) of the form

$$(G, K) = (G[Z, \lambda], K^{\circ}[Z]) \quad \text{or} \quad (G, K) = (G[Z, \lambda], K^{\otimes}[Z]).$$

REMARK. It is also possible to consider intermediate wreath products K between $K^{\circ}[Z]$ and $K^{\otimes}[Z]$, below we consider one example from this zoo.

2.8. Colors, smells, melodies. We wish to draw figures, also we want to have more flexible language.

a) We assign to each Ω_j a *color*, say, red, blue, white, red, green, etc. We also think that a color is an attribute of all points of Ω_j . We denote colors by \mathfrak{I}_j .

b) Next, we assign to each \mathbb{N}_i a *smell* \mathbb{N}_i , say, Magnolia, Matricaria, Pinus, Ledum, Rafflesia, etc. On figures we denote smells by $\blacktriangle, \mathbf{x}, \blacksquare, \dots$. We also think that a smell \mathbb{N}_i is an attribute of all points of $(\mathbb{N}_i \times \mathbb{I}(\zeta_{ji})) \subset \Omega_j$.

c) Orbits of a group $\mathbb{S}_{\infty}(\mathbb{N}_i)$ on Ω_j are one-point orbits or countable homogeneous spaces $\mathbb{S}_{\infty}/\mathbb{S}_{\infty}^1 \simeq \mathbb{N}$. We assign to each countable orbit a *melody*, say, violin, harp, tomtom, flute, drum, \dots . On figures we draw melodies by symbols $\heartsuit, \succ, \nabla, \sharp, \ddagger$, etc. Note that a melody makes sense after fixation of a smell and a color.

EXAMPLE. See Figure 1.c. We have a 4×3 table. Rows are distinguished by colors, columns are separated by smells. Rows inside each box are numerated by melodies. \square

2.9. Admissible representations. Let ρ be a unitary representation of G . We say that ρ is a *K-admissible representation* if the restriction of ρ to K is admissible. Equivalently, we say that ρ is a *representation of the pair* (G, K) .

2.10. Reformulation of admissibility in terms of continuity. The embedding $K \rightarrow G$ admits an extension to the map $\mathbf{K} \rightarrow \mathbf{G}$ of the corresponding completions. Consider the group $G \cdot \mathbf{K}$ generated by G and \mathbf{K} ,

$$G \subset G \cdot \mathbf{K} \subset \mathbf{G}.$$

Any element of $G \cdot \mathbf{K}$ admits a (non-unique) representation as $g\mathbf{k}$, where $g \in G$, $\mathbf{k} \in \mathbf{K}$.

We consider the natural topology on the subgroup \mathbf{K} and assume that \mathbf{K} is an open-closed subgroup in $G \cdot \mathbf{K}$.

Proposition 2.3 *A unitary representation of G is K-admissible if and only if it is continuous in the above sense.*

PROOF. Let ρ be an admissible representation of G in a Hilbert space H . We define the action of \mathbf{K} as a continuous extension of the action of K . \square

2.11. Lemma on admissibility.

Lemma 2.4 *Let ρ be an irreducible unitary representation of G . If some $H^\alpha \neq 0$, then the representation ρ is K -admissible.*

PROOF. Consider the subspace $\mathcal{H} := \cup H^\alpha$. Fix $g \in G$. For sufficiently large β , the element g commutes with K^β . Therefore \mathcal{H} is g -invariant. The closure of \mathcal{H} is a subrepresentation. \square

Corollary 2.5 *If an irreducible unitary representation of G has a K -fixed vector, then it is K -admissible.*

2.12. Existing representation theory. The most interesting object of existing theory is the pair $G = \mathbb{S}_\infty \times \mathbb{S}_\infty$, $K = \mathbb{S}_\infty$ is the diagonal subgroup, [31], [25], [11]). In our notation $Z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The representation theory of this pair includes also earlier works on Thoma characters (see [35], [36], [37]).

Olshanski [31] also considered pairs:

$$- G = \mathbb{S}_{\infty+1} \times \mathbb{S}_\infty, K = \mathbb{S}_\infty; \text{ in our notation, } Z = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \Lambda = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$- G = \mathbb{S}_{2\infty}, K = \mathbb{S}_\infty \times \mathbb{S}_\infty; \text{ in our notation } Z = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

- $G = \mathbb{S}_{2\infty}$, $K = \mathbb{S}_\infty \ltimes \mathbb{Z}_2^\infty$ and also $G = \mathbb{S}_{2\infty+1}$ with the same subgroup K . In our notation, $Z = (2)$ and $\Lambda = 0$ or 1 .

In all these cases, pairs (G, K) are limits of spherical pairs of finite groups.

The author in [19] considered the case $G = \mathbb{S}_\infty \times \cdots \times \mathbb{S}_\infty$ with the diagonal subgroup $K = \mathbb{S}_\infty$. In our notation, $Z = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

2.13. Train. Consider (see Figure 2) the following matrix $\Theta_N^{[\alpha]} \in \mathbb{S}_\infty$ of the size $(\alpha + N + N + \infty) \times (\alpha + N + N + \infty)$,

$$\Theta_N^{[\alpha]} = \begin{pmatrix} 1_\alpha & 0 & 0 & 0 \\ 0 & 0 & 1_N & 0 \\ 0 & 1_N & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \in \mathbb{S}_\infty.$$

In fact, $\Theta_N^{[\alpha]}$ is contained in K^α .

Consider a pair (G, K) . For a multi-index $\alpha = (\alpha_1, \dots, \alpha_p)$ we denote by $\Theta_N^{[\alpha]}$ the element

$$\Theta_N^{[\alpha]} = \left(\Theta_N^{[\alpha_1]}, \dots, \Theta_N^{[\alpha_p]} \right) \in K$$

Again, $\Theta_N^{[\alpha]} \in K^\alpha$.

Fix multi-indices α, β, γ . Consider double cosets

$$\mathfrak{h} \in K^\beta \backslash G/K^\alpha, \quad \mathfrak{g} \in K^\gamma \backslash G/K^\beta,$$

and choose their representatives $g \in \mathfrak{g}$, $h \in \mathfrak{h}$. Consider the sequence

$$f_N = g\Theta_N^{[\alpha]}h \in G.$$

Consider the double coset \mathfrak{f}_N containing f_N ,

$$\mathfrak{f}_N \in K^\gamma \backslash G/K^\alpha.$$

Theorem 2.6 a) *The sequence \mathfrak{f}_N is eventually constant.*

b) *The limit*

$$\mathfrak{g} \circ \mathfrak{h} := \lim_{N \rightarrow \infty} \mathfrak{f}_N$$

does not depend on a choice of representatives g, h .

c) *The product \circ obtained in this way is associative, i.e., for any*

$$\mathfrak{g} \in K^\delta \backslash G/K^\gamma, \quad \mathfrak{h} \in K^\gamma \backslash G/K^\beta, \quad \mathfrak{f} \in K^\beta \backslash G/K^\alpha,$$

we have

$$(\mathfrak{g} \circ \mathfrak{h}) \circ \mathfrak{f} = \mathfrak{g} \circ (\mathfrak{h} \circ \mathfrak{f}).$$

Thus we obtain a category $\mathbb{T}(G, K)$, whose objects are multiindices α and morphisms $\alpha \rightarrow \beta$ are elements of $K^\beta \backslash G/K^\alpha$. We say that $\mathbb{T}(G, K)$ is the *train* of the pair (G, K) .

2.14. Involution in the train. The map $g \mapsto g^{-1}$ induces a map of quotient spaces $K^\alpha \backslash G/K^\beta \rightarrow K^\beta \backslash G/K^\alpha$, we denote it by

$$\mathfrak{g} \mapsto \mathfrak{g}^\square.$$

Evidently,

$$(\mathfrak{g} \circ \mathfrak{h})^\square = \mathfrak{h}^\square \mathfrak{g}^\square.$$

2.15. Representations of the train. Now let ρ be a unitary representation of the pair (G, K) . We define subspaces H^α as above, denote by P^α the operator of orthogonal projection to H^α . For $\mathfrak{g} \in K^\beta \backslash G/K^\alpha$, choose its representative $g \in \mathfrak{g}$. Consider the operator

$$\bar{\rho}_{\alpha, \beta}(g) := P^\beta \rho(g) : H^\alpha \rightarrow H^\beta.$$

By definition, we have

$$\|\bar{\rho}_{\alpha, \beta}(g)\| \leq 1. \tag{2.1}$$

Theorem 2.7 a) *An operator $\bar{\rho}_{\alpha, \beta}(g)$ depends only on a double coset \mathfrak{g} containing g .*

b) *We get a representation of the category $\mathbb{T}(G, K)$, i.e., for any*

$$\mathfrak{g} \in K^\gamma \backslash G/K^\beta, \quad \mathfrak{h} \in K^\beta \backslash G/K^\alpha$$

the following identity holds

$$\bar{\rho}_{\beta,\gamma}(\mathfrak{g})\bar{\rho}_{\alpha,\beta}(\mathfrak{h}) = \bar{\rho}_{\alpha,\gamma}(\mathfrak{g} \circ \mathfrak{h}).$$

c) We get a $*$ -representation. i.e.,

$$(\bar{\rho}_{\alpha,\beta}(\mathfrak{g}))^* = \bar{\rho}_{\beta,\alpha}(\mathfrak{g}^\square).$$

d) $\rho(\Theta_N^\alpha)$ weakly converges to the projection P^α .

Theorem 2.8 *Our construction provides a bijection between the set of all unitary representation of the pair (G, K) and the set of all $*$ -representations of the category $\mathbb{T}(G, K)$ satisfying the condition (2.1).*

We omit proofs of Theorems 2.6–2.8 and Theorem 2.9 formulated below, because proofs are literal copies of proofs in [21].

Our main purpose is to give explicit description of trains, also we give some constructions of representations of groups.

2.16. Sphericity.

Theorem 2.9 *Consider a pair $(G, K) = (G[Z, \Lambda], K^\circ[Z])$, or $(G[Z, \Lambda], K^\oplus[Z])$ as above. If $\Lambda = 0$, then the pair (G, K) is spherical. In other words, for any irreducible unitary representation of (G, K) the dimension of the space of K -fixed vectors is ≤ 1 .*

2.17. Further structure of the paper. In Sections 8, 9 we present the description of trains for arbitrary pairs

$$(G(Z, \Lambda), K^\circ[Z]), \quad (G(Z, \Lambda), K^\oplus[Z]).$$

We also present examples of representations and spherical functions.

I am afraid that it is difficult to understand constructions in such generality. For this reason, in Sections 3–7 we consider simple special cases: pairs (G, K) connected with matrices

$$Z = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} 2 & \dots & 2 \end{pmatrix}, \quad Z = \begin{pmatrix} 3 \end{pmatrix}.$$

Well-representative examples are contained in Section 7.

Now we give 4 examples (details are below), to outline the language used for description of double coset spaces.

EXAMPLE 1. Take collection of beads of n colors. We say that a chaplet is a cyclic sequence of $2k$ beads, we also equip beads of a chaplets by interlacing signs \pm . Consider the set Ξ , whose elements are (non-ordered) collections of chaplets. Then $\Xi \simeq K^\circ[Z] \setminus G[Z, 0]/K^\circ[Z]$, where $Z = \begin{pmatrix} 2 & \dots & 2 \end{pmatrix}$.

EXAMPLE 2. Consider the set Δ of (generally, disconnected) oriented triangulated closed two-dimensional surfaces with additional data: triangles are colored in black and white checker-wise. We also forbid a triangulation of

a sphere into two triangles. Let $(Z) = 3$, $\Lambda = 0$. We consider the subgroup $K^\odot := \mathbb{S}_\infty \ltimes (\mathbb{Z}_3)^\infty$ intermediate between $K^\circ[Z]$ and $K^\otimes[Z]$. The set Δ is in a canonical one-to-one correspondence with the double coset space $K^\odot \setminus G[Z, 0]/K^\odot$.

EXAMPLE 3. Consider the set Φ of all tri-valent graphs, whose vertices are colored black and white (and neighboring vertices have different colors). We forbid components having only two vertices. The set Φ is in one-to-one correspondence with $\Xi \simeq K^\circ[Z] \setminus G[Z, 0]/K^\circ[Z]$, where $Z = (3)$.

EXAMPLE 4. (Belyi data) Consider the Riemannian sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ with 3 distinguished points $0, 1, \infty$. Consider the set Ψ all (generally, disconnected) ramified coverings $\Gamma \rightarrow \overline{\mathbb{C}}$ with 3 branching points $0, 1, \infty$ (requiring that the covering is nontrivial on each component of Γ). Then Ψ is in a canonical one-to-one correspondence with the set $K^\otimes[Z] \setminus G[Z, 0]/K^\otimes[Z]$, where $Z := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

2.18. Remarks on generality. Note, that our generality is not maximal.

1. Our constructions can be easily extended to the case $p = \infty$ or $q = \infty$.
2. However, it seems that number of melodies of a fixed color and a fixed smell must be finite. Also, in our picture, orbits of $\mathbb{S}_\infty(\mathbb{N}_i)$ on Ω_j are fixed points or $\mathbb{S}_\infty(\mathbb{N}_i)/\mathbb{S}_\infty^1(\mathbb{N}_i)$. It seems that this is necessary, otherwise we have embeddings $\mathbb{S}_\infty(\mathbb{N}_i) \rightarrow \mathbb{S}_\infty(\Omega_j)$ and not to $\mathbb{S}_\infty(\Omega_j)$.
3. Let $L, M \subset G[Z, \Lambda]$ be subgroups of the type described above and $L, M \supset K^\otimes[Z]$. Then double cosets $L \setminus G[Z, \Lambda]/M$ admit a description on the language of Section 8.

3 First example. Symmetric group and Young subgroup

3.1. Group. Now $q = 1$ and p is arbitrary,

$$Z = (1 \ \dots \ 1), \quad \lambda \geq 0 \text{ is arbitrary.}$$

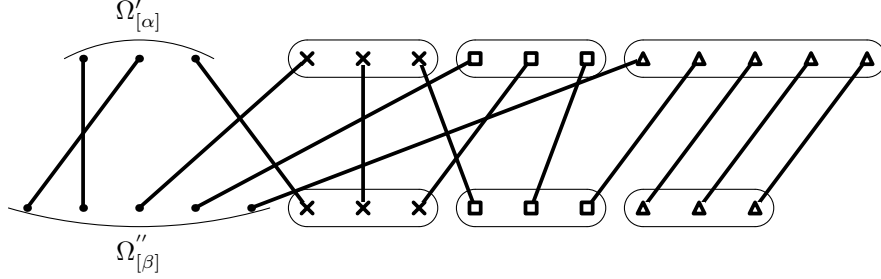
Therefore, $\Omega = \mathbb{N}_1 \sqcup \dots \sqcup \mathbb{N}_p \sqcup L$, and

$$\begin{aligned} (G, K) &= (G[Z, \Lambda], K^\circ[Z]) = (G[Z, \Lambda], K^\otimes[Z]) = \\ &= (\mathbb{S}_\infty(\Omega), \mathbb{S}_\infty(\mathbb{N}_1) \times \dots \times \mathbb{S}_\infty(\mathbb{N}_p)). \end{aligned}$$

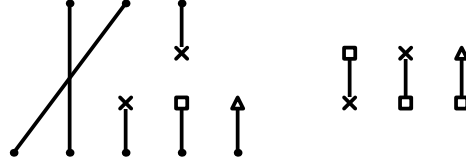
We assign a smell \aleph_i to each \mathbb{N}_i . On some figures we draw symbols $\mathbf{x}, \blacksquare, \blacktriangle$ instead of smells.

3.2. Description of the category. For a subgroup $K^\alpha \subset K$ denote by $\Omega_{[\alpha]} \subset \Omega$ the subset fixed by K^α ,

$$\Omega_{[\alpha]} = L \sqcup \mathbb{M}_1(\alpha_1) \sqcup \dots \sqcup \mathbb{M}_p(\alpha_p).$$



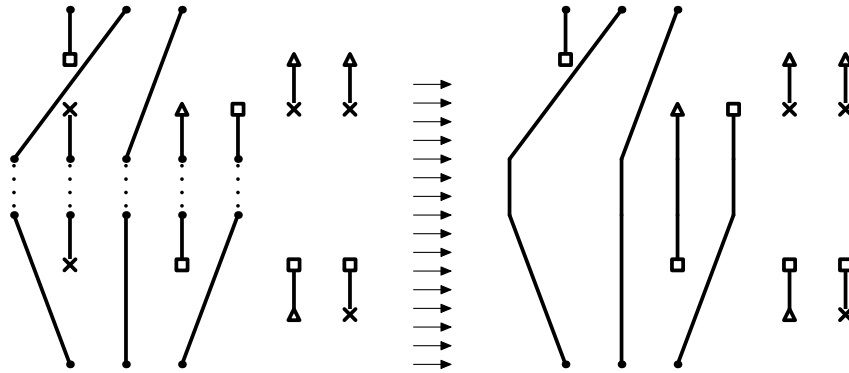
a) We present two copies of the set Ω . An element $g \in \mathbb{S}_\infty(\Omega)$ is drawn as a collection of segments connecting ω' and $(g\omega)''$. The set Ω' (respectively Ω'') is split into the pieces in a natural way. The first piece is $\Omega'_{[\alpha]}$, any point of the remain is contained in some set \mathbb{N}_i and has the corresponding smell. We draw smells as \times , \square , Δ .



b) We forget part of information about segments. We remember

1. ends that are contained in $\Omega'_{[\alpha]}$, $\Omega''_{[\beta]}$.
2. smells of remaining ends.
3. direction up/down of segments.

Segments whose ends have the same smell are removed.



c) Two diagrams (left figure) and their product (right figure).

Figure 3: Reference to Section 3.

Consider two copies Ω' and Ω'' of the set Ω . For a point of $\omega \in \Omega$ we denote its copies by ω' and ω'' .

For any $g \in \mathbb{S}_\infty(\Omega)$ we construct an oriented one-dimensional manifold $\Xi_{\alpha,\beta}[g]$ with boundary (in fact, $\Xi_{\alpha,\beta}[g]$ is a union of segments). It is easier to look to Figure 3, but we present a formal description.

a) If $\omega \in \Omega_{[\alpha]}$, $g\omega \in \Omega_{[\beta]}$, we draw a segment, whose origin is ω' and the end is $(g\omega)''$.

b) If $\omega \in \Omega_{[\alpha]}$, $g\omega \notin \Omega_{[\beta]}$, then we draw a segment starting in ω' and mark another end of the segment by the smell of $g\omega$.

c) Let $\omega \notin \Omega_{[\alpha]}$, $g\omega \in \Omega_{[\beta]}$. Then we draw a segment, mark its origin by the smell of ω , another end of the segment is $(g\omega)''$.

d) Let $\omega \notin \Omega_{[\alpha]}$, $g\omega \notin \Omega_{[\beta]}$. If smells of ω and $g\omega$ coincide, then we do not draw a segment.

e) Let $\omega \notin \Omega_{[\alpha]}$, $g\omega \notin \Omega_{[\beta]}$. If smells of ω and $g\omega$ are different, we draw a segment whose origin has the smell of ω and the end has the smell of $g\omega$.

We say that points of $\Omega'_{[\alpha]}$ are *entries* of $\Xi_{\alpha,\beta}(g)$ and points of $\Omega''_{[\beta]}$ are *exits*.

Due d) we get only finite collection of segments.

Lemma 3.1 *If g_1, g_2 are contained in the same element of $K^\beta \setminus G/K^\alpha$, then $\Xi_{\alpha,\beta}[g_1] = \Xi_{\alpha,\beta}[g_2]$.*

This is obvious. □

Now consider two elements $\mathfrak{g} \in K^\beta \setminus G/K^\alpha$, $\mathfrak{h} \in K^\gamma \setminus G/K^\beta$. To obtain the diagram corresponding $\mathfrak{h} \circ \mathfrak{g}$, we identify each exit of $\Xi_{\alpha,\beta}(\mathfrak{g})$ with the corresponding entry of $\Xi_{\beta,\gamma}(\mathfrak{h})$ and get a new oriented manifold. It remains to remove segments whose origin and end have the same smell, see Fig.3.c).

Proposition 3.2 *This product is the product in the train $\mathbb{T}(G, K)$ of the pair (G, K) .*

PROOF. This is obvious, see Fig. 2. □

3.3. The involution in the category. We change entries and exits, and reverse orientations of the segments.

4 Tensor products of Hilbert spaces

4.1. Definition of tensor products. Let H_1, H_2, \dots be a countable collection of Hilbert spaces (they can be finite-dimensional or infinite-dimensional). Fix a unit vector $\xi_k \in H_k$ in each space. The tensor product

$$(H_1, \xi_1) \otimes (H_2, \xi_2) \otimes (H_3, \xi_3) \otimes \dots$$

is defined in the following way. We choose an orthonormal basis $e_j[k]$ in each H_k , assuming $e_1[k] = \xi_k$. Next, we consider the Hilbert space with orthonormal basis

$$e_{\alpha_1}[1] \otimes e_{\alpha_2}[2] \otimes \dots$$

such that $e_{\alpha_N}[N] = \xi_N$ for sufficiently large N (note, that this basis is countable).

Distinguished vectors are necessary for the definition (otherwise we get a non-separable object).

Construction essentially depends on a choice of distinguished vectors. The spaces $\otimes(H_k, \xi_k)$ and $\otimes(H_k, \eta_k)$ are canonically isomorphic if and only if

$$\sum |\langle \xi_j, \eta_j \rangle - 1| < \infty.$$

In particular, we can omit distinguished vectors in a finite number of factors (more precisely we can choose them in arbitrary way).

All subsequent definitions are standard (see the paper of von Neumann [33] with watching of all details or short introduction in [7]).

4.2. Action of symmetric groups in tensor products. The symmetric groups S_n act in tensor powers $H^{\otimes n}$ by permutations of factors. This phenomenon has a straightforward analog.

We denote by

$$(H, \xi)^{\otimes \infty} := (H, \xi) \otimes (H, \xi) \otimes \dots$$

the infinite symmetric power of (H, ξ) .

Proposition 4.1 a) *The complete symmetric group S_∞ acts in $(H, \xi)^{\otimes \infty}$ by permutation of factors. The representation is continuous with respect to the topology of S_∞ .*

b) *The vector $\xi^{\otimes \infty}$ is a unique S_∞ -fixed vector in $(H, \xi)^{\otimes \infty}$.*

c) *The subspace of S_∞^α -fixed vectors is*

$$H^{\otimes \alpha} \otimes \xi^{\otimes \infty}.$$

Proposition 4.2 *Fix a sequence ξ_k of unit vectors in a Hilbert space H . The symmetric group S_∞ acts in the tensor product $\otimes_k(H, \xi_k)$ by permutations of factors.*

Emphasis that in this case there is no action of the complete symmetric group S_∞ .

4.3. First application to (G, K) -pairs. Now consider the same objects Ω, G, K , as in the previous section, recall that

$$\Omega = L \sqcup \mathbb{N}_1 \sqcup \dots \sqcup \mathbb{N}_p.$$

Consider a Euclidean space H of dimension $\leq p$ and a collection of unit vectors $\eta_1, \dots, \eta_p \in H$ generating H . Consider a countable collection of copies

H_ω of H enumerated by elements of the set Ω . For each $\omega \in \Omega$ we choose an element ξ_ω according the following rule:

- if $\omega \in \mathbb{N}_i$, then $\xi_\omega = \eta_i$,
- if $\omega \in L$, then we choose a unit vector ξ_ω in an arbitrary way.

Consider the tensor product

$$\mathcal{H} := \bigotimes_{\omega \in \Omega} (H, \xi_\omega) = \bigotimes_{\omega \in L} H \otimes \bigotimes_{i=1}^p (H, \eta_i)^{\otimes \infty}.$$

The group $G = \mathbb{S}_\infty(\Omega)$ acts in \mathcal{H} by permutation of factors by Proposition 4.2.

Each group $\mathbb{S}_\infty(\mathbb{N}_i)$ acts in \mathcal{H} by permutations of factors in

$$(H, \eta_i)^{\otimes \infty}$$

Therefore, we get the action of $\mathbf{K} = \prod_{i=1}^p \mathbf{S}_\infty(\mathbb{N}_i)$ in \mathcal{H} .

Thus, the group $G \cdot \mathbf{K}$ acts in the tensor product \mathcal{H} by permutations of factors.

The parameter determining such representations is the Gram matrix

$$a_{kl} = \langle \eta_k, \eta_l \rangle$$

of the system η_i .

4.4. Super-tensor products (a subsection for experts in representation theory). Below we construct representation for arbitrary (G, K) -pairs related to symmetric groups. All constructions described below can be extended to super-tensor products as it is explained in [31], [19].

For the (G, K) -pair discussed now this gives nothing.

5 Chips

Construction of this section is an extension of Olshanski [31]. We use the term “chip” following Kerov [12], note that the first version of chips³ was introduced by R. Brauer in [5].

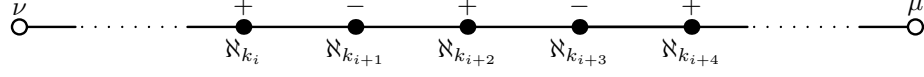
5.1. The group. Take $Z = (2 \ \dots \ 2)$, let Λ be arbitrary. Consider the pair $(G, K) = (G[Z, \Lambda], K^\circ[Z])$. Now

$$\Omega = L \sqcup \prod_{i=1}^p (\mathbb{N}_i \times \mathbb{I}(2))$$

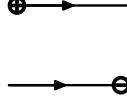
and K is a product of hyperoctohedral groups.

$$K = \prod_{i=1}^p (\mathbb{S}_\infty(\mathbb{N}_i) \ltimes \mathbb{Z}_2^\infty).$$

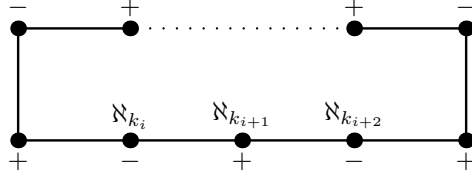
³A matching on the union of two sets.



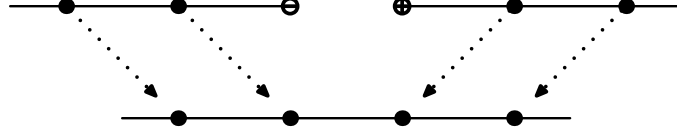
a) An open chain.



b) Ends of an open chain.



c) A closed chain.



d) Gluing of two chains.

Figure 4: Reference to Section 5.

By $\omega \mapsto \omega^\circ$ we denote the natural involution in Ω , it acts by the transposition of two elements of $\mathbb{I}(2)$, this involution fixes points of the set L .

For each \mathbb{N}_i , we attribute a smell \mathbb{N}_i , say $\mathbf{x}, \mathbf{q}, \mathbf{\Delta}$, etc.

5.2. Description of the train. For a subgroup K^α denote by $\Omega_{[\alpha]}$ the set of all point fixed by K^α ,

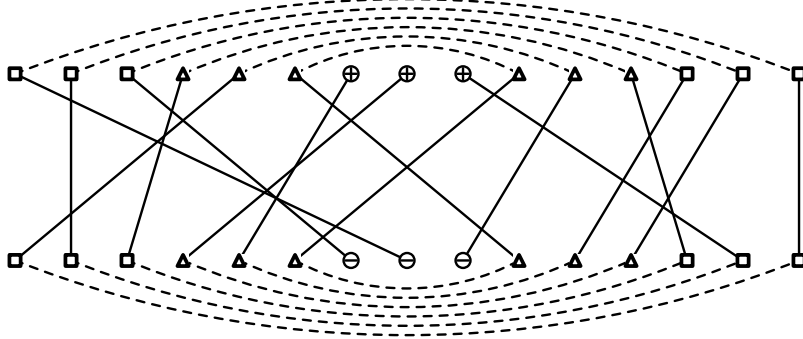
$$\Omega_{[\alpha]} := L \sqcup \prod_{i=1}^p (\mathbb{M}_i(\alpha_i) \times \mathbb{I}(2)).$$

Such sets are objects of the train $\mathbb{T}(G, K)$.

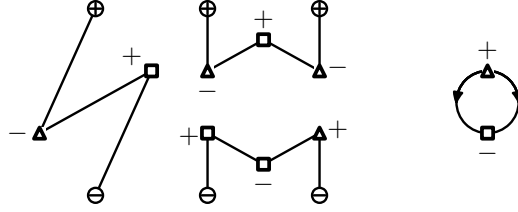
We wish to assign a graph with additional data to any double coset in $K^\beta \backslash G/K_\alpha$. The construction is explained on Figure 5. However, we repeat this formally.

Consider two copies of Ω , say Ω' and Ω'' . Consider the 'set of entries' $\Omega'_{[\alpha]}$ and the 'set of exits' $\Omega''_{[\beta]}$.

a) A *closed chain* is a finite cyclic sequence of distinguished points (vertices)



a) An element g of $\mathbb{S}_\infty(\Omega)$ for $p = 2$. Elements of subsets $\Omega'_{[\alpha]}$ are marked by \oplus , elements of $\Omega''_{[\beta]}$ by \ominus . Horizontal arcs mark the involution in Ω .



b) The diagram corresponding to the element g . We contract horizontal arcs to a vertex and remember smells of these arcs (symbols \square , \blacktriangle). We also set a symbol “+” (respectively, “−”), which bears in mind was an arc upper or lower.

Figure 5: Reference to Section 4.

and segments connecting neighboring vertices. A vertex has two labels: a smell \aleph_{k_j} and \pm . Symbols $+$ and $-$ interlace (therefore a length of a closed chain is even and there are only two possible choices of signs).

b) An *open chain* has similar properties. Interior vertices are labeled by smells and \pm . Ends are points of the set $\Omega'_{[\alpha]} \cup \Omega''_{[\beta]}$. If an end is an ‘entry’, i.e., an element of $\Omega'_{[\alpha]}$, we mark it by \oplus . If an end is an exit, then we mark it by \ominus . Pluses and minuses interlace.

If both ends are ‘entries’ (or both are ‘exits’), then the length of the chain is even. If one of the ends is an entry and another one is an exit, then the length is odd. Note that there is a unique possible choice of signs if we fixed types of ends.

To define the product of morphisms $\mathfrak{G} : \Omega_{[\alpha]} \rightarrow \Omega_{[\beta]}$, $\mathfrak{H} : \Omega_{[\beta]} \rightarrow \Omega_{[\gamma]}$, we identify exits of the diagram \mathfrak{G} with the corresponding entries of \mathfrak{H} . Points of gluing become interior points of edges, see Figure 6.

5.3. Construction of a diagram from an element of \mathbb{S}_∞ . For each $\omega \in \Omega$, we draw a segment with vertices ω' and $(g\omega)''$.

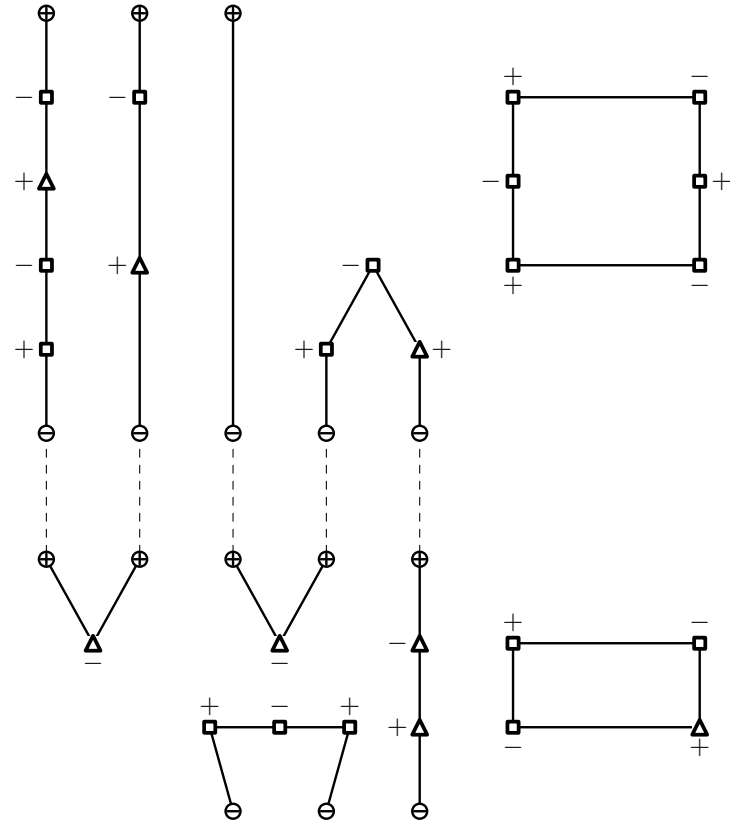


Figure 6: Reference to Subsection 6.2. Two chips and their product. We connect exits of the upper diagram with entries of the lower diagram.

1. We identify ω' with $(\omega^\circ)'$ and ω'' with $(g\omega^\circ)''$. Thus we get a countable disjoint union of finite chains. Endpoints of open chains are elements of $\Omega'_{[\alpha]}$ and $\Omega''_{[\beta]}$.

2. We forget point ω , ω'' corresponding to interior points of chains and remember only their smells.

3. We remove cycles of length 2 having vertices of the same smell.

See Figure 5.

5.4. Examples of representations. For a Hilbert space H denote by

$$SH^k \subset H^{\otimes k}$$

its symmetric power.

Consider a Hilbert space V , and collection of unit vectors

$$\xi_i \in S^2V \subset V \otimes V,$$

where $i = 1, \dots, p$. Consider the tensor product

$$\bigotimes_{i=1}^p \left(\bigotimes_{\omega \in \mathbb{N}_i} (V \otimes V, \xi_i) \right) \otimes \bigotimes_{\lambda \in L} V$$

The group $G \cdot \mathbf{K}$ acts by permutations of factors V .

The parameter of a representation is a collection of vectors $\xi_i \in S^2V$ defined up to action of the complete unitary group $U(V)$ of the Hilbert space V .

Note that for $p = 1$ (the case considered by Olshanski [31]) we have a unique vector ξ and it can be reduced to a diagonal form.

6 Chips (continuation)

Here we discuss the same group G and change the subgroup. Also, we change language (pass from a graph to a dual graph).

6.1. Group. Let $Z = (2 \dots 2)$. Consider the same objects \mathbb{N}_j , Ω , L , as in Subsection 5.1. Consider the pair

$$(G, K) = (G[Z, \Lambda], K^{\otimes}[Z]),$$

i.e.,

$$K = \prod_{i=1}^p \mathbb{S}_\infty(\mathbb{N}_i)$$

Now for each \mathbb{N}_i we have two embeddings $\mathbb{N}_i \rightarrow \Omega$, which are distinguished by their melodies.

Consider the same sets $\Omega_{[\alpha]}$ as in Subsection 5.1.

The difference is following. In the previous section ends of horizontal arcs in Fig.5.a had equal rights. Now they have different melodies. Since there are

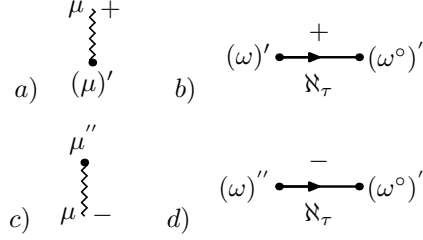


Figure 7: Reference to Subsection 6.2. Items of a chip

only two melodies, we prefer to think that arcs are oriented and arrows look from the left to right (for instance, we think that origins of arcs are 'violins' and ends are 'contrabass's'). Now we contract *vertical* (i.e., inclined) arcs and get a graph composed of chains and cycles.

We explain the construction more carefully.

6.2. Construction of the graph. Consider the following items for building a graph, see Figure 7.

a) Let $\mu \in \Omega_{[\alpha]}$. We draw a vertex and a tail as it is shown on Figure 7. We write μ and “+” on the tail. Also we write a label μ' on the vertex.

b) Let $\omega \notin \Omega_{[\alpha]}$. For definiteness, let the melody of ω be 'violin', therefore the melody of ω° be 'contrabass'. Then we draw an oriented segment with the label ω' on the origin and the label $(\omega^\circ)'$ on the end. We mark these segments by the smell of ω (it coincides with the smell of ω°). We also write the label “+” on the segment.

c) Let $\nu \in \Omega_{[\beta]}$. We draw a vertex and a tail. We write μ and “−” on the tail. Also we set a label ν'' on the vertex.

d) Let $\omega \notin \Omega_{[\beta]}$. Let the melody of ω be 'violin'. Then we draw an oriented segment with the label ω'' on the origin and the label $(\omega^\circ)''$ on the end. We mark these vertices by the smell of ω . We also write the label “−” on the segment.

Thus we get a collection of items. For each $\omega \in \Omega$ we identify the vertex with label $(\omega)'$ and the vertex with label $(\omega)''$.

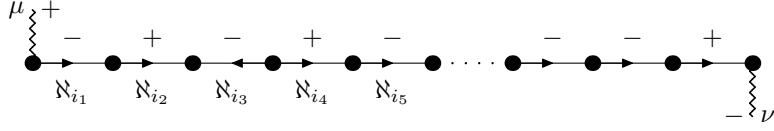
Next, we forget labels on vertices.

We get collection of chains of two types (see Figure 8).

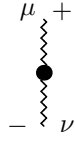
a) *An open chain.* It is a chain of oriented segments, each segment has a smell and a sign. An end of a chain is a tail (equipped with a label μ and a sign). Signs interlace.

We say that end vertices with “+” tails are *entries*, and vertices with “−” tails are *exits*.

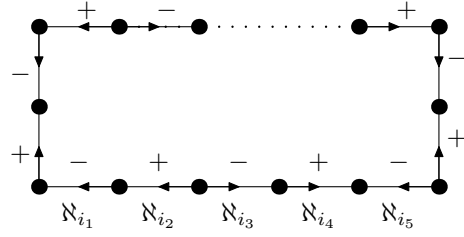
b) *A closed chain* consists of oriented segments equipped with smells and signs. Signs interlace.



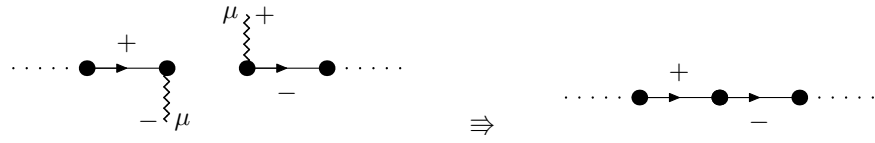
a) An open chain.



b) A short open chain.



c) A closed chain.



d) Gluing of chains.

Figure 8: Reference to Section 6.

It remains to define the multiplication of two chips $\mathfrak{G} : \Omega_{[\alpha]} \rightarrow \Omega_{[\beta]}$, $\mathfrak{h} : \Omega_{[\beta]} \rightarrow \Omega_{[\gamma]}$. We identify exits of \mathfrak{G} with corresponding entries of \mathfrak{h} and cut off tails.

Proposition 6.1 *This multiplication coincides with the multiplication in the train of (G, K) .*

It seems that this is self-obvious, see Figure 2. \square

6.3. Examples of representations. We repeat the construction of Subsection 5.4. Now we can choose distinguished vectors $\xi_j \in V \otimes V$ in arbitrary way (not necessary $\xi_j \in SV^2$).

7 Example: triangulated surfaces

7.1. Group. Let $Z = (3)$, $\lambda \geq 0$ be arbitrary. First, we consider the pair

$$(G(Z, \Lambda), K^\circ(Z)) = (\mathbb{S}_{\lambda+3\infty}, \mathbb{S}_\infty \ltimes (S_3)^\infty).$$

We reduce the subgroup and assume

$$K := \mathbb{S}_\infty \ltimes (\mathbb{Z}_3)^\infty,$$

where $\mathbb{Z}_3 \subset S_3$ is the group the group of cyclic permutations (or, equivalently, the group of even permutations).

Now

$$\Omega = L \sqcup (\mathbb{N} \times \mathbb{I}(3)).$$

Let $\alpha \geq 0$, the set $\Omega_{[\alpha]}$ is defined as above (as fixed points of K^α),

$$\Omega_{[\alpha]} = L \sqcup (\mathbb{M}(\alpha) \times \mathbb{I}(3)).$$

Thus we have only one color, only one smell, but 3 melodies, say harp (∇), violin (\heartsuit), tube (\succ).

7.2. Encoding of elements of symmetric group. Fix $\alpha, \beta \geq 0$.

First, we take the following collection of items (see Figure 9).

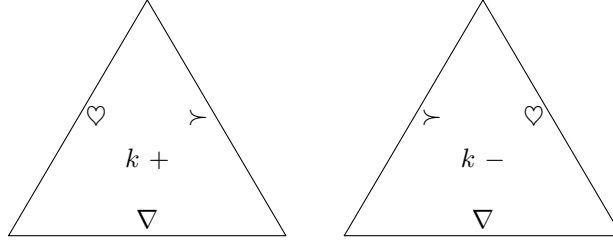
A. *Plus-triangles and minus triangles.* For each element of $k \in \mathbb{N}$ we draw a pair of oriented triangles $T_\pm(k)$ with label k . We write labels $\nabla, \heartsuit, \succ$ on the sides $T_+(k)$ (resp. $T_-(k)$) clock-wise (resp. anti-clock-wise).

An important remark: a number k and a melody determines some element of Ω .

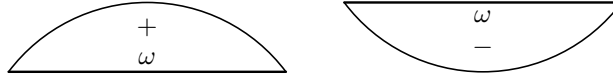
B. *Plus-tags and minus tags.* For each element $\omega \in \Omega$ we draw two oriented segments $D_\pm(\omega)$ with tags, see Figure 9. We write the label ω and label '++' (respectively '--') on the segment $T_+(\omega)$ (resp. $T_-(\omega)$).

In this way, we have produced too much items. Next, we remove all

- triangles $T_+(k)$ with $k \leq \alpha$;
- triangles $T_-(k)$ with $k \leq \beta$;



a) A plus-triangle and a minus-triangle.



b) A plus-tag and a minus-tag.

Figure 9: Reference to Section 7. Items for a complex.

— tags $D_+(\omega)$, where $\omega \notin \Omega_{[\alpha]}$;

— tags $D_+(\omega)$, where $\omega \notin \Omega_{[\beta]}$.

Now each element of Ω is present on precisely one edge of one item $T_+(k)$ or $D_+(\omega)$ (and respectively on one item $T_-(k)$ or $D_-(\omega)$).

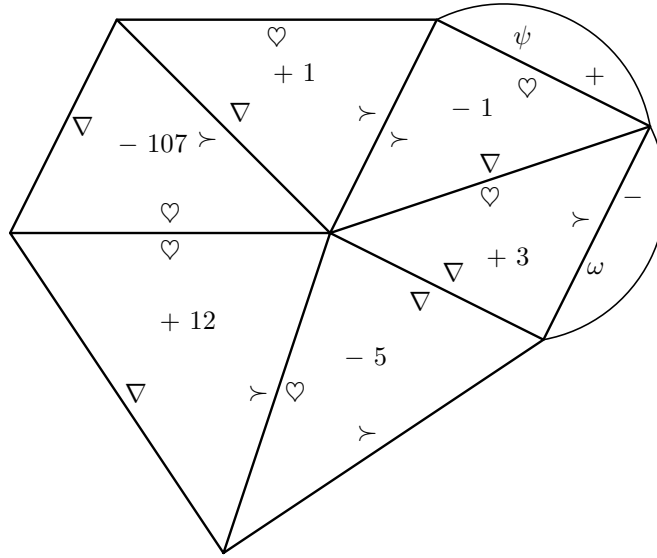
Fix element $g \in \mathbb{S}_\infty(\Omega)$. For each $\omega \in \Omega$ we identify (keeping in mind the orientations) the edge of $T_+(\cdot)$ or $D_+(\cdot)$ labeled by ω with the edge of $T_-(\cdot)$ or $D_-(\cdot)$ labeled by $g\omega$.

In this way, we get a two-dimensional oriented triangulated surface $\Xi(g)$ with tags on the boundary. Our picture satisfies the following properties:

- (i) A surface consists of a countable number of compact components.
- (ii) Each component is a two-dimensional oriented triangulated surface with tags on the boundary (we admit also a segment with two tags, see Figure 11.a).
- (iii) All triangles have labels '+' or '-', neighboring triangles have different signs.
- (iv) Plus-triangles (resp. minus-triangles) are enumerated by $\alpha + 1, \alpha + 2, \alpha + 3, \dots$ (resp. $\beta + 1, \beta + 2, \beta + 3, \dots$).
- (v) Sides of plus-triangles are labeled (from interior) by $\nabla, \heartsuit, \succ$ clockwise. Sides of minus-triangles are labeled by the same symbols anti-clockwise.
- (vi) Tags are labeled by \pm . Plus-tags are enumerated by elements of $\Omega_{[\alpha]}$, minus-tags by elements of $\Omega_{[\beta]}$.
- (vii) Almost all components are spheres composed of two triangles and labels on sides of the triangle coincide.

We consider such surfaces up isotopes preserving the orientation.

Lemma 7.1 *Each surface equipped with data given above has the form $\Xi[g]$. Different $g \in \mathbb{S}_\infty(\Omega)$ produce different equipped surfaces.*



A piece of complex. Removing numbers and melodies \succ , ∇ , \heartsuit (and leaving signs and labels on tags), we pass to double cosets.

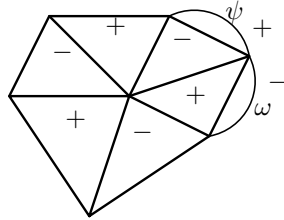
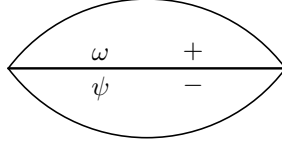
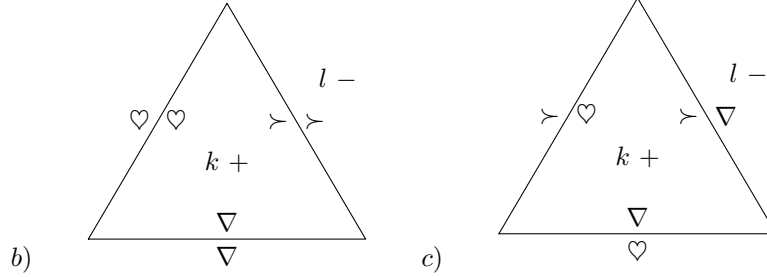


Figure 10: Reference to Section 7.



a) A degenerate component. An edge with two tags.



b) This is a stereographic projection of a sphere and a graph on a sphere. A pure envelope (b) and an envelope (c).

Figure 11: Reference to Section 6.

PROOF. We present the inverse construction. Above we have assigned two elements of Ω to each edge. Let μ correspond to the plus-side, ν corresponds to the minus-side. Then g send μ to ν . \square

Thus we get a bijection.

Now we need a technical definition. We say that an *envelope* is a component consisting of two triangles. We say that an envelope is *pure* if melodies on both sides of each edge coincide (see Figure 11).

7.3. Projection to double cosets.

Lemma 7.2 *Right multiplications $g \mapsto gh$ by elements of $\mathbb{S}_\infty^\alpha(\mathbb{N})$ correspond to permutations of labels $\alpha + 1, \alpha + 2, \alpha + 3, \dots$ on plus-triangles. Respectively left multiplication by elements of $\mathbb{S}_\infty^\beta(\mathbb{N})$ correspond to permutations of labels on minus-triangles.*

Lemma 7.3 *Right multiplications by elements of the group $(\mathbb{Z}_3)^{-\alpha+\infty} \subset K^\alpha$ correspond to cyclic permutations of symbols $\nabla, \heartsuit, \succ$ inside each plus-triangle.*

Corollary 7.4 *Pass to double cosets $K^\beta \backslash G / K^\alpha$ corresponds to forgetting numbers of triangles, melodies of sides, and removing all envelopes.*

Thus, we remember only labels on tags and signs.

7.4. Construction of the train. Objects of the category are $\alpha \geq 0$. Fix indices α and β . A *morphism* $\alpha \rightarrow \beta$ is a compact (generally, disconnected) triangulated surface without envelopes equipped with data (iii), (vi) from the list above (labels \pm and labels on tags).

To multiply $\mathfrak{G} : \alpha \rightarrow \beta$, $\mathfrak{H} : \beta \rightarrow \gamma$, we glue (according the orientations) minus-segments of the boundary of \mathfrak{G} with plus-segments of the boundary of \mathfrak{H} having the same labels. We remove corresponding tags, forget their labels, forget the contour of gluing. It can appear some envelops, we remove them.

We get a morphism $\alpha \rightarrow \gamma$.

Theorem 7.5 *The multiplication described above is the multiplication in the train of the pair (G, K) .*

PROOF. Fix $h, g \in G$. Consider the corresponding $\Xi[h], \Xi[g]$. Take very large N . Then the set of labels k on minus-triangles of $\Xi[\theta_\beta^N h]$ and the set of labels l on plus-triangles of $\Xi[g]$ are disjoint. Therefore plus-triangles of $\Xi[\theta_\beta^N h]$ preserve their neighbors after the multiplication $\Xi[\theta_\beta^N h] \rightarrow g\Xi[\theta_\beta^N h]$. Also, minus-triangles of $\Xi[g]$ preserve their neighbors after multiplication $g \mapsto g\Xi[\theta_\beta^N h]$. Therefore both surfaces $\Xi[h], \Xi[g]$ are pieces of the surface $\Xi[g\theta_\beta^N h]$. \square

7.5. Involution on the train. We reverse signs and reverse the orientation.

7.6. Examples of representations. Let V be a Hilbert space. Fix unit vectors

$$\xi_1, \xi_2, \xi_3 \in V \otimes V \otimes V \quad (7.1)$$

invariant with respect to cyclic permutations of elements of the tensor products.

Consider the tensor product

$$\bigotimes_{l \in L} V \otimes \bigotimes_{i=1}^3 (V \otimes V \otimes V, \xi_i)^{\otimes \infty}.$$

The group $G \cdot \mathbf{K}$ acts on this product by permutations of factors.

7.7. Another pair. Let $Z, \lambda, G = G[Z, \lambda]$ be the same, consider the pair

$$(G, K) := (G(Z, \lambda), K^{\otimes}(Z)) = (\mathbb{S}_{\lambda+3\infty}, \mathbb{S}_{\infty}).$$

Return to Lemma 7.2. Now we remove numbers of triangles but preserve melodies. We also remove all pure envelops.

In construction of representation, we can replace (7.1) by arbitrary unit vectors

$$\xi_1, \xi_2, \xi_3 \in V \otimes V \otimes V.$$

7.8. Another pair. Let $Z, \lambda, G = G[Z, \lambda]$ be the same. Consider the pair

$$(G, K) := (G(Z, \lambda), K^{\circ}(Z)) = (\mathbb{S}_{\lambda+3\infty}, \mathbb{S}_{\infty} \ltimes (S_3)^{\infty}).$$

First, we construct representations. In the construction of Subsection 7.6 we take

$$\xi_1, \xi_2, \xi_3 \in S^3 V \subset V \otimes V \otimes V.$$

An attempt to repeat the construction of the train meets an obvious difficulty: permutations of melodies change orientations of triangles. However, we can pass from triangulations to dual graphs. Now we can enumerate double cosets by tri-valent graphs. See the following section.

8 General case, $K = K^\circ$ is a wreath product

8.1. Group. Here we consider an arbitrary matrix Z and an arbitrary vector Λ . Now

$$\begin{aligned}\Omega_j &= L_j \sqcup \prod_{i=1}^p (\mathbb{N}_i \times \mathbb{I}(\zeta_{ji})), \\ G &:= G[Z, \Lambda] = \prod_{j=1}^q \mathbb{S}_\infty(\Omega_j), \\ K &:= K^\circ[Z] = \prod_{i=1}^p \left(\mathbb{S}_\infty(\mathbb{N}_i) \ltimes \left(\prod_j S_{\zeta_{ji}} \right)^\infty \right)\end{aligned}$$

Recall that we have attributed a color to each Ω_j , a smell to each \mathbb{N}_i , and a melody to each infinite orbit of $\mathbb{S}_\infty(\mathbb{N}_i)$ on Ω_j .

We denote

$$\Omega := \prod_{j \leq q} \Omega_j$$

and regard $G[Z, \lambda]$ as a subgroup in $\mathbb{S}_\infty(\Omega)$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_p)$ we denote by $\Omega_{[\alpha]}$ the set of all K^α -fixed points of Ω ,

$$\Omega_{[\alpha]} = \prod_{j=1}^q \left(L_j \sqcup \prod_{i=1}^p (\mathbb{M}(\alpha_i) \times \mathbb{I}(\zeta_{ji})) \right).$$

8.2. Encoding of elements of symmetric groups. For each element of $G[Z, \lambda]$ we construct a graph equipped with some additional data.

For each smell i we draw a *node* $T[\mathbb{N}_i]$ (see Figure fig:graph). It contains a vertex of smell \mathbb{N}_i and $\sum_j \zeta_{ji}$ semi-edges. Edges are colored, each color \mathfrak{J}_j is used for coloring ζ_{ji} edges. Also we attribute a melody to each semi-edge. Thus semi-edges of $T[\mathbb{N}_i]$ are in one-to-one correspondence with orbits of $\mathbb{S}_\infty(\mathbb{N}_i)$ on Ω .

Now we prepare the following collection of items.

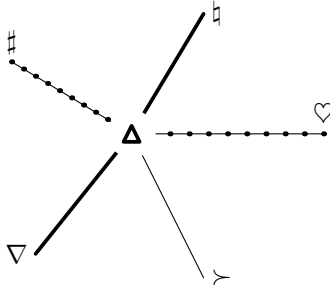
- a) For each smell i and each $k \in \mathbb{N}$ we draw two copies $T_\pm[\mathbb{N}_i; k]$ of the node $T[\mathbb{N}_i]$, their vertices are labeled k and \pm . We through out nodes $T_+[\mathbb{N}_i; k]$ with $k \leq \alpha_i$ and $T_+[\mathbb{N}_i; m]$ with $m \leq \beta_i$
- b) For each color j and for each element ω of $\Omega_j \cap \Omega_{[\alpha]}$ we draw a tag $D_+(\omega)$ and mark this tag by ω , the color of ω , and " + ". We draw similar tags $D_-(\omega)$ for elements $\omega \in \Omega_{[\beta]}$. We imagine a tag as a vertex and semi-edge.

Thus the set Ω is in one-to-one correspondence with the sets

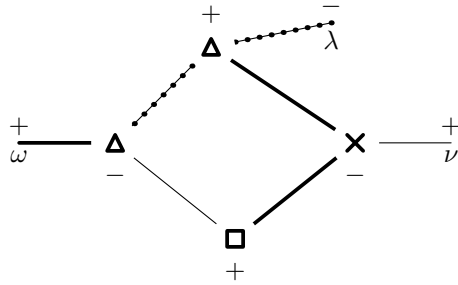
$$\mathcal{E}_+ = \left\{ \begin{array}{l} \text{All semi-edges of} \\ \text{all nodes } T_+[\mathbb{N}_i, k] \end{array} \right\} \bigcup \Omega_{[\alpha]}$$

and

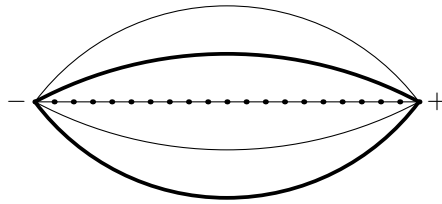
$$\mathcal{E}_- = \left\{ \begin{array}{l} \text{All semi-edges of} \\ \text{all nodes } T_-[\mathbb{N}_i, k] \end{array} \right\} \bigcup \Omega_{[\beta]}.$$



a) A node.



b) A double coset.



c) A *trivial* component of a graph.

Figure 12: Reference to Subsection 8.2.

Denote the bijections $\Omega \rightarrow \mathcal{E}_\pm$ by H_\pm .

Now for each $\omega \in \Omega$ we connect a semi-edge $H_+(\omega) \in \mathcal{E}_+$ with the semi-edge $H_-(g\omega) \in \mathcal{E}_-$. We get a graph with following properties.

- (i) Graph consists of a countable number of compact components.
- (ii) There are 2 types of vertices, interior vertices⁴ and end-vertices (ends of semi-edges).
- (iii) Each interior vertex has a smell \aleph_i and a sign " + " or " - ".
- (iv) Interior plus-vertices are enumerated by the set $\{\alpha+1, \alpha+2, \dots\}$, interior minus-vertices by $\{\beta+1, \beta+2, \dots\}$.
- (v) End-vertices fall into two classes, entries and exits. Entries are enumerated by elements of $\Omega_{[\alpha]}$ and labels " + ". Exits are enumerated by elements of $\Omega_{[\beta]}$ and labels " - ".
- (v) Neighboring vertices have different signs.
- (vi) Edges are colored, number of edges of a color \beth_j coming to an interior vertex of smell \aleph_i is ζ_{ji} . The edge adjacent to an end vertex with label ω has the color of ω .
- (vii) For each semi-edge adjacent to an interior vertex it is attributed a melody compatible with its color. At each vertex of smell \aleph_i each compatible with the smell melody is present precisely one time.
- (viii) All but finite number of components consists of two interior vertices and edges connected this vertices.

We call components described in (viii) *trivial*. We say that a component is *completely trivial* if for each edge smells of both semi-edges coincide.

Theorem 8.1 *There is one-to-one correspondence between the set of all graphs satisfying (i)-(viii) and the infinite symmetric group*

PROOF. Consider an edge. It has a plus-semi-edge and a minus-semi-edge. Consider the corresponding elements $\varphi \in \mathcal{E}_+$ and $\psi \in \mathcal{E}_-$. We set $g\varphi = \psi$. \square

8.3. Projection to double cosets.

Proposition 8.2 a) *Right multiplications by elements of $\mathbb{S}_\infty^\alpha(\aleph_i)$ correspond to permutations of labels $\{\alpha+1, \alpha+2, \dots\}$ on plus-vertices of the smell \aleph_i .*

b) *Right multiplications by elements of $S_{\zeta_{ij}}^{-\alpha_i+\infty} \subset \mathbb{S}_\infty(\Omega_j)$ correspond to permutations of melodies of semi-edges of color \beth_j adjacent to fixed vertices of the smell \aleph_i .*

Corollary 8.3 *Projection to double cosets correspond to forgetting labels $\in \mathbb{N}$ and melodies*

⁴The case $\sum_j \zeta_{ji} = 1$ is admissible, then we can meet a unique edge in an interior vertex of the smell \aleph_i

Colors, smells, signs, and also labels on tags are preserved.

8.4. Multiplication of double cosets. For two morphisms $\mathfrak{G} : \alpha \rightarrow \beta$, $\mathfrak{H} : \beta \rightarrow \gamma$, we glue exits of \mathfrak{g} with corresponding entries of \mathfrak{h} .

The involution is the inversion of signs and also entries/exits.

Theorem 8.4 *This product coincides with the product in the train $\mathbb{T}(G, K)$.*

PROOF. See proof of Theorem 7.5.

8.5. Some representations of (G, K) . We consider a collection of Hilbert spaces W_1, \dots, W_q enumerated by colors. Fix i . Consider the tensor product

$$\mathcal{H}_i = \bigotimes_{j=1}^p W_j^{\otimes \zeta_{ji}}. \quad (8.1)$$

Observation 8.5 *Factors of the product are in one-to-one correspondence with semi-edges $a \in T[\aleph_i]$*

Fix a unit vector

$$\xi_i \in \bigotimes_{j=1}^p S^{\zeta_{ji}} W_j \subset \mathcal{H}_i. \quad (8.2)$$

Consider the tensor product

$$\mathfrak{W} := \bigotimes_{j=1}^q W_j^{\otimes \lambda_j} \otimes \bigotimes_{i=1}^q (\mathcal{H}_i, \xi_i)^{\otimes \infty}.$$

Note that factors W of this tensor product are enumerated by elements of $\sqcup \Omega_j$. Formally we can write

$$\bigotimes_{j=1}^q \bigotimes_{\omega \in \Omega_j} W_j$$

However, this makes no sense without distinguished vectors.

Each group $\mathbb{S}_\infty(\Omega_j)$ acts by permutations of factors W_j . This determines the action of G .

The groups $\mathbb{S}_\infty(\aleph_i)$ act as permutations of factors in

$$(\mathcal{H}_i, \xi_i)^{\otimes \infty}.$$

Thus we get action of $G \cdot \mathbf{K}$.

8.6. Reduction of the subgroup. Now consider the pair $(G, K) = (G[Z, \Lambda], K^\otimes[Z])$. We use the same encoding of $G[Z, \Lambda]$.

For passing to double cosets $K^\beta \setminus G/K^\alpha$ we forget numeric labels (but remember melodies). Also we remove completely trivial components of the graph.

In the following section we give another (equivalent) description of double cosets.

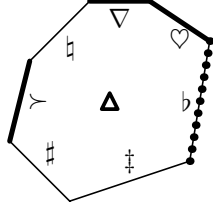


Figure 13: Reference to Section 7. A polygon $T_+[N_i]$.

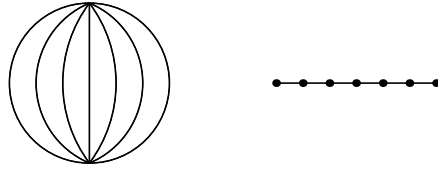


Figure 14: Reference to Subsection 9.5. A di-gonal complex and the corresponding one-dimensional chain.

9 General case, K is a product of symmetric groups

Construction of this section is more-or less a version of the previous construction. For smaller group K we can replace a graph by a fat graph and after this draw a two dimensional surface. We repeat the construction independently.

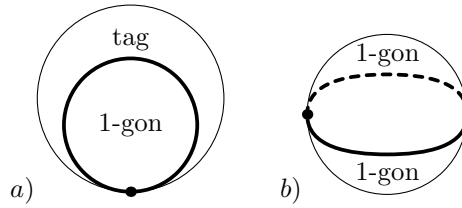


Figure 15: Reference to Section 9.5. The only possible connected 1-gonal complexes.

9.1. Group. We consider an arbitrary matrix Z and an arbitrary vector Λ . Now

$$\begin{aligned}\Omega_j &= L_j \sqcup \prod_{i=1}^p (\mathbb{N}_i \times \mathbb{I}(\zeta_{ji})), \\ G &:= G[Z, \Lambda] = \prod_{j=1}^q \mathbb{S}_\infty(\Omega_j), \\ K &:= K^\otimes[Z] = \prod_{i=1}^p \mathbb{S}_\infty(\mathbb{N}_i).\end{aligned}$$

Recall that we have attributed a color to each Ω_j , a smell to each \mathbb{N}_i , and a melody to each infinite orbit of $\mathbb{S}_\infty(\mathbb{N}_i)$ on Ω_j .

We denote

$$\Omega := \prod_{j \leq q} \Omega_j.$$

For a multiindex $\alpha = (\alpha_1, \dots, \alpha_p)$ we denote by $\Omega_{[\alpha]}$ the set of all K^α -fixed points of Ω ,

$$\Omega_{[\alpha]} = \prod_{j=1}^q \left(L_j \sqcup \prod_{i=1}^p (\mathbb{M}(\alpha_i) \times \mathbb{I}(\zeta_{ji})) \right).$$

9.2. Constructions of the train. Fix a smell \mathbb{N}_i . Nontrivial orbits of $\mathbb{S}_\infty(\mathbb{N}_i)$ on $\sqcup \Omega_j$ are enumerated by a pair (color, melody). Total number of such orbits is

$$\sum_j \zeta_{ij}.$$

We choose an arbitrary cyclic order on the set of such pairs (the construction below depends on this choice). Next, we draw a polygon $T_+[\mathbb{N}_i]$ of smell \mathbb{N}_i , whose sides are marked by pairs (color, melody) according the cyclic order. We also define the polygon $T_-[\mathbb{N}_i]$, whose sides are marked according the reversed cyclic order.

REMARK. Let us forget melodies on the sides of the polygon. If there is no rotation of the polygon preserving the coloring of sides, then melodies can be reconstructed in a unique way. \square

Consider the following collection of items.

— *Plus-polygons and minus-polygons.* For each $i \leq p$ for each $k \in \mathbb{N}_i$ we draw the pair of oriented polygons $T_\pm[\mathbb{N}_i, k]$ that were described above; they are additionally labeled by $k \in \mathbb{N}$. Any side of any polygon $T_\pm[\mathbb{N}_i, k]$ has smell, color, and melody; therefore a side determines an element of Ω .

— *Plus-tags and minus-tags.* For each element $\omega \in \Omega$, we draw two tags $D_\pm(\omega)$ labeled by ω and \pm , see Figure 9. The side of a tag is painted in the color of ω

Fix multi-indices α, β .

We remove some items from the collection:

- polygons $T_+[\aleph_i, k]$ if $k \leq \alpha_i$;
- polygons $T_-[\aleph_i, m]$ if $m \leq \beta_i$;
- tags $T_+[\omega]$ if $\omega \notin \Omega_+[\alpha]$;
- tags $T_-[\omega]$ if $\omega \notin \Omega_-[\beta]$.

Now we have the one-to-one correspondences between the set Ω and set of all edges of all plus-triangles and plus-tags. Also we have one-to-one correspondences between the set Ω and set of all edges of all minus-triangles and minus-tags.

For each $g \in G$ we glue the complex. For each $\omega \in \Omega$, we identify the (oriented) edge of a plus-polygon or a plus-tag corresponding ω with the (oriented) edge of a minus-polygon or minus-tag corresponding $g\omega$.

Thus we get a polygonal two-dimensional oriented surface with tags on the boundary satisfying the following properties:

- (i) The surface consists of countable number of compact components.
- (ii) Each component is tiled by polygons of the types $T_\pm[\aleph_i]$ and has tags D_\pm on the boundary.
- (iii) Each polygon is labeled by '+' or '-', neighboring polygons have different signs.
- (iv) Each edge has a color, which is common for both (plus and minus sides of an edge).
- (v) Each edge has two melodies, on the plus-side and on the minus-side.
- (vi) Cyclic order of pairs (color, melody) around the perimeter of each polygon $T_\pm[\aleph_i]$ is fixed.
- (vii) Plus-polygons (resp., minus-polygons) of a fixed smell \aleph_i are enumerated by $\alpha + 1, \alpha + 2, \dots$ (resp. $\beta + 1, \beta_2, \dots$).
- (viii) Plus-tags are enumerated by points of $\Omega_{[\alpha]}$ and minus-tags by points of $\Omega_{[\beta]}$.
- (ix) All but a finite number of components of the surface are unions of pairs $T_+[\aleph_j, k]$ and $T_-[\aleph_j, l]$. We call such components '*envelopes*'. We say that a *pure envelope* is an envelope such that melodies on plus and minus sides of each edge coincide.

Theorem 9.1 *Data of such type are in one-to-one correspondence with the group G .*

Inverse construction. To find $g\omega$, we find ω inside pairs (plus-polygon, side). This side also is a side of minus-polygon and codes the element $g\omega$.

9.3. Pass to double cosets $K^\beta \backslash G / K^\alpha$. The literal analog of Lemma 7.2 holds.

To pass to double cosets $K^\alpha \backslash G / K^\beta$ we forget labels $k \in \mathbb{N}$ and remove all envelopes.

We get a compact surface tiled by polygons;

— Polygons are equipped with signs \pm and smells,

— Each edge is equipped with a color and a pair of melodies on the negative side and the positive side of the edge (coloring and melodization of edges of each polygon is fixed up to a cyclic permutation of sides as above)⁵.

— The boundary edges of the surface are equipped with signs, positive edges are enumerated by points of $\Omega_{[\alpha]}$, negative edges by points of $\Omega_{[\beta]}$.

We say that such surface is a morphism $\alpha \rightarrow \beta$.

Let $\mathfrak{G} : \alpha \rightarrow \beta$, $\mathfrak{H} : \beta \rightarrow \gamma$ be two surfaces. For each $\omega \in \Omega[\beta]$ we glue ω -exit of \mathfrak{G} with ω -entry of \mathfrak{H} (according the orientation). Removing envelops, we come to a complex of the same type.

Theorem 9.2 *This multiplication coincides with the multiplication in the train,*

PROOF is the same as for Theorem 7.5.

9.4. Involution in the train. We reverse sign \pm and reverse the orientation.

9.5. Simple cases. Note that our construction admits 2-gons and 1-gons. Let the matrix Z satisfies $\sum_j \zeta_{ij} = 2$ for all i . Then all our polygons are 2-gons. A digonal complex can be regarded as union of chains and we come to chip-type constructions (see Sections 4, 6).

If $Z = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}$, then our complex consists of 1-gons. There are only 3 types of possible components and we come to construction of Section 3.

9.6. Belyi data, see [3], [4], [14], [38]. Consider the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$, compact closed Riemannian surface Γ and a covering map $\Gamma \rightarrow \overline{\mathbb{C}}$ having ramifications only at points $0, 1, \infty \in \overline{\mathbb{C}}$. We draw intervals $[0, 1]$, $[1, \infty]$, $[\infty, 0]$ on the sphere and paint them in red, yellow, blue respectively. We set the label '+' on the upper half-plane in $\overline{\mathbb{C}}$ and '-' on the lower half-plane.

Lift this picture to the covering Γ . We get a triangulation of Γ , triangles are labeled by \pm and edges are colored. In our notation this corresponds to double cosets

$$K^{\otimes}[Z] \setminus G[Z, \Lambda] / K^{\otimes}[Z] \quad \text{with } Z = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \Lambda = 0$$

Belyi Theorem claims that such maps $\Gamma \rightarrow \overline{\mathbb{C}}$ exists if and only if a curve Γ is determined over algebraic closure of \mathbb{Q} .

9.7. Constructions of representations. In the construction of tensor product from Subsection 8.5 we can choose arbitrary unit vectors

$$\xi_i \in \bigotimes_{j=1}^p W_j^{\otimes \zeta_{ji}} =: \mathcal{H}_i.$$

instead of (8.2).

⁵Recall that in many cases melodies can be uniquely reconstructed from colors and may be forgotten

10 Spherical functions

We wish to write explicit formulas for spherical functions of some representations in terms of trains. To be definite, we consider pairs $(G, K) = (G[Z, 0], K^{\otimes}[Z])$ and a representation ρ of G in the space

$$\mathfrak{W} := \bigotimes_{i=1}^q (\mathcal{H}_i, \xi_i)^{\otimes \infty}$$

described above. The formula written below is a precise copy of [19]. We omit a proof.

10.1. Formula. We consider a morphism $\mathfrak{G} : 0 \rightarrow 0$, i.e. a closed polygonal complex $\mathfrak{G} : \alpha \rightarrow \beta$.

Choose an orthonormal basis $e_k[j]$ in each W_j .

Now we assign elements $e_k[j]$ to edges of the complex according the following condition: colors of $e_k[j]$ correspond to colors of edges.

Look to a polygon. Keeping in the mind Observation 8.5, we observe that a tensor product of basis vectors e along the perimeter of the polygon P is a basis vector, in \mathcal{H} , say $E_{\Phi}(P)$. The formula for spherical function is

$$\sum_{\Phi} \prod_{\text{plus-polygons } P} \langle E_{\Phi}(P), \xi_{\aleph(P)} \rangle_{\mathcal{H}} \cdot \prod_{\text{minus-polygons } Q} \langle \xi_{\aleph(Q)}, E_{\Phi}(Q) \rangle_{\mathcal{H}} \quad (10.1)$$

where $\aleph(P)$ is the smell of P and the summation is given over all arrangements of basis vectors.

REMARK. For chips this formula is reduced to calculations of traces of products of matrices.

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